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The Solenoid and Warsawanoid Are Sharkovskii Spaces

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ABSTRACT

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We extend Sharkovskii's theorem concerning orbit lengths of endomorphisms of the real line to endomorphisms of a path component of the solenoid and certain subspaces of the Warsawanoid. In particular, Sharkovskii showed that if there exists an orbit of length 3 then there exist orbits of all lengths. The solenoid is the inverse limit of double covers over the circle, and the Warsawanoid is the inverse limit of double covers over the Warsaw circle. We show Sharkovskii's result is true for path components of the solenoid and certain subspaces of the Warsawanoid.

Keywords: Sharkovskii theorem, covering spaces, solenoid, Warsawanoid, inverse limit, Warsaw circle

Acknowledgments

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Contents

Chapter 1. Introduction

[1, 1-3] Sharkovskii's Theorem is a well-known result in dynamical systems. It is named after Oleksandr Mikolaiovich Sharkovskii, a prominent Ukrainian mathematician, who submitted a paper titled *Coexistence of cycles of a continuous mapping of a line into itself* to the Ukrainian Mathematical Journal in 1962. The paper was published by the journal in 1964. The paper provided a proof to the following theorem: *If a continuous mapping of the reals into the reals has a point with fundamental period k, and if $k < l$ with respect to a specific special ordering, then the mapping also has a point with fundamental period l.*

Despite its current popularity and use in many areas of Mathematics, the paper and its theorem received very little recognition and prestige outside of Eastern Europe until the late 1970's. There are several potential reasons for this. First and foremost, the paper was originally written in Russian and published in a Soviet journal. It is also possible that the field of dynamical systems theory did not become fashionable until later. At any rate, it was not until Tien-Yien Li and James Yorke published a famous paper titled *Period three implies chaos* in 1975 that Sharkovskii's work became well-known outside Eastern Europe.

1.1 3 Implies Chaos

[2] Li and Yorke were interested in mathematically modeling the evolution of natural processes and phenomena, a branch of mathematics called dynamical systems. Of course, many natural systems and phenomena can be modeled with differential equations or difference equations, but they were interested in more simplistic situations where a system is evolving through discrete states such that the nature of the system in each state can be expressed by one number. What's more, they required that the number x_{i+1} describing the system in state $i + 1$ be obtainable by a continuous endomorphism f on an interval of the reals, such

that $f(x_i) = x_{i+1}$ where x_i is the number describing the state of the system in state i. Thus, Li and Yorke were concerned with repeated iterations of a continuous endomorphism on a real interval. This type of model provides a method to model population growth, the spread of disease, financial markets, and many more situations of interest to researchers within pure and applied mathematics.

This paper played a very influential role in the growth of dynamical systems theory by sparking widespread interest in the field and its applications. There were many results in the paper but one of the more famous ones was the following:

Theorem: ***3 implies chaos.*** Let $J \subset \mathbb{R}$ be an interval, and let $f : J \to J$ be a continuous function. If there exists a point $x \in J$ such that $f^3(x) = x$, and $f^n(x) \neq x$ for $n \in \{1, 2\}$, then for each integer $m \in \mathbb{N}$, there exists a point $x_m \in J$ such that $f^m(x) = x$ and $f^k(x) \neq x$ for all $l \in \{1, 2, \ldots, k-1\}$.

This result was groundbreaking; as such, the paper gained popularity, and Li and Yorke traveled to many conferences lecturing on their work. At one particular conference in East Berlin, they met Sharkovskii, who pointed out that the result *3 implies chaos* is a special case of a more general theorem he had proved more than a decade earlier. This let to worldwide recognition of Sharkovskii's work.

1.2 Reworking the Theorem

[1, 1-3] It did not take long for the study of dynamical systems and chaos theory to increase in popularity and to spread knowledge of Sharkovskii's Theorem. As such, many mathematicians worked to find shorter and simpler proofs of the Theorem. By 1980, three elegant proofs of the theorem, all similar, were published by such prominent mathematicians as

Guckenheimer, Block, Young and Misiurewicz, Morris and Ho, and Burkart. These proofs relied extensively on the Intermediate Value Theorem and have become the "standard" proofs.

1.3 Dynamical Systems and Sharkovskii Today

Today, dynamical systems has become a central field of study in modern mathematics and has allured the interest and work of some of the world's brightest mathematicians. Sharkovskii's theorem is well-known as one of the field's foundational and integral theorems. In fact, Sharkovskii is honored as one of few mathematicians alive today with his name attached to one of his results.

1.4 Generalizing Sharkovskii's Theorem

Since 1975, mathematicians have been seeking other topological spaces on which continuous endomorphisms satisfy the conclusion of Sharkovskii's theorem; we refer to such spaces as *Sharkovskii spaces*. [3] In 1980, Block, Guckenheimer, Misiurewicz, and Young published a paper showing that S^1, the one-dimensional sphere, is a Sharkovskii space. [4, 164] In 1985, Helga Schirmer first defined a Sharkovskii space as we have done here and proved that an ordered topological space Y is Sharkovskii if and only Y is ordered densely and Y has the least upper bound property for every subset of Y bounded above. [21] In 2012 Grant, Conner, and Meilstrup published a paper with the title *A Sharkovsky Theorem for non-locally Connected Spaces* showing that the following spaces are Sharkovskii: the topologist's sine curve, any n-fold union of topologist sine curves, the Warsaw circle, any n-fold cover of the Warsaw circle, and any line of topologist sine curves.

Even for spaces that are not Sharkovskii, much work has been done in studying periods of orbits of self-maps. For work done on S^1, reference [5] [6] [7]; for n-ods, reference [8] [9] [10]; for trees, reference [11] [12]; for the figure-eight space, reference [13]; for further work on Warsaw circle and k-Warsaw circle, reference [14] [15]; for hereditarily decomposable chainable continua, reference [16].

It is worth noting that all the Sharkovskii spaces mentioned are one-dimensional, since the theorem easily fails for many higher dimensional spaces. For example, rotating a two-dimensional disk by angle $\frac{2\pi}{3}$ is a clear counterexample. Thus, many weaker versions of the theorem have been attempted for higher dimensional spaces, but none have gained the widespread fame as the original theorem.

In this paper we provide two more Sharkovskii spaces: the inverse limit of double covers over the circle, which we call the Solenoid, and an inverse limit of double covers over the Warsaw circle, which we call the Warsawanoid.

Many mathematicians are still working to provide a classification of all Sharkovskii spaces.

1.5 Sharkovskii's Theorem

Before stating the theorem, we give a few definitions.

Definition 1.1.1: [17, 229-231] We define a new ordering of the Natural Numbers called the *Sharkovskii Ordering*.

$$
\begin{aligned}
&3 < 5 < 7 < 9 < 11 < \dots \\
&< 2(3) < 2(5) < 2(7) < 2(9) < 2(11) < \dots \\
&< 2^2(3) < 2^2(5) < 2^2(7) < 2^2(9) < 2^2(11) < \dots \\
&< 2^3(3) < 2^3(5) < 2^3(7) < 2^3(9) < 2^3(11) < \dots \\
&\qquad\qquad\vdots \\
&\dots < 2^4 < 2^3 < 2^2 < 2 < 1
\end{aligned}
$$

Definition 1.1.2: Let f be a continuous function from an interval $I \subseteq \mathbb{R}$ to itself (the interval need not be open or closed). Denote by f^n the nth composition of f with itself. Let $x \in I$. If $f^n(x) = x$ and $f^k(x) \neq x$ for all $k \in \mathbb{N}$, $1 \leq k < n$, we say that x has *orbit* n. If there exists an x with orbit n in the domain of f, we say that f has an n-*orbit*.

Theorem: ***Sharkovskii.*** Let f be a continuous function from an interval $I \subseteq \mathbb{R}$ to itself, where I need not be closed or open. If f has an n-orbit, then f has an m-orbit for all $m \geq n$ with respect to the Sharkovskii Ordering.

1.6 Generalizing the Theorem to the Solenoid and Warsawanoid

The purpose of this paper is to extend the theorem to continuous endomorphisms on the *Solenoid*, the inverse limit of double covers over the circle and the *Warsawanoid*, the inverse limit of double covers over the Warsaw Circle.

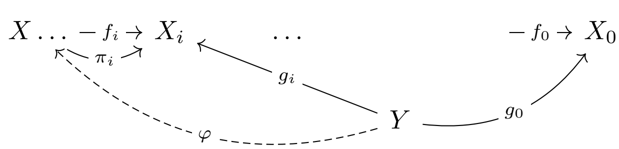
Definition 1.1.3: Let $f : X \to X$ be a map on a space X. If the orbits of f satisfy the conclusion of Sharkovskii's Theorem, we say that f has the *Sharkovskii Property*. If every map $f : X \to X$ has the Sharkovskii Property, we say X is a Sharkovskii Space.

Definition 1.1.4: [18, 2-3] Given topological spaces X_i and connecting maps $f_i : X_{i+1} \to X_i$,

$$\dots \xrightarrow{f_2} X_2 \xrightarrow{f_1} X_1 \xrightarrow{f_0} X_0$$

the inverse limit is defined to be the unique subspace $\{(\dots x_2, x_1, x_0)\}|\ x_i \in X_i$ and $f_i(x_{i+1}) = x_i$ for all i $\}$ of the product space. Inverse limits come with canonical projections π_i from the inverse limit to each X_i, by sending a coherent sequence to its i-th coordinate. This is a fact we will use extensively throughout this paper.

Inverse limits have the following *Universal Mapping Property*. If Y is a space and $g_i : Y \to X_i$ are maps satisfying $g_i = f_i \circ g_{i+1}$ for all i, then there exists a unique map φ from Y to the inverse limit, X, making the diagram below commute.



Definition 2.1.1: For $n \in \mathbb{N} \cup \{0\}$, let C_n be the unit circle lying in the complex plane. Let $g_n : C_{n+1} \to C_n$ defined by $g_n(x) = x^2$ be the connecting maps. We define the inverse limit of this system to be The Solenoid, denoted throughout this chapter by S. The projection maps from S to C_n we denote by $\pi_n : S \to C_n$.

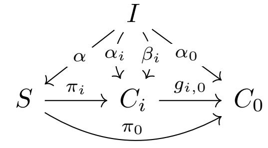
2.1 Properties of the Solenoid

We develop a few important relationships between $\mathbb{R}$ and S.

Theorem 2.1.1: The point $(\ldots p_2, p_1, p_0)$ is in the same path component of S as the point $(\ldots 1, 1, 1)$, if and only if the sequence of real numbers $\{|2^n a(p_n)|\}$ is bounded as n goes to infinity, where $a(p_n) \in [-\pi, \pi)$ is the argument of the complex number p_n.

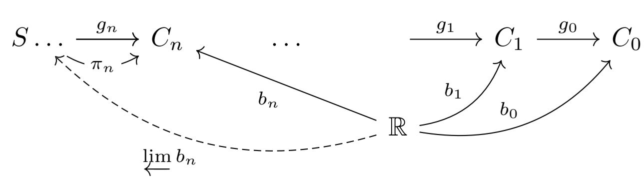
Proof: Recall that the projection maps from S to C_n are $\pi_n : S \to C_n$, and the connecting maps are $g_n : C_{n+1} \to C_n$ defined by $g_n(x) = x^2$. Then we have the compatible maps $g_{i,j} : C_i \to C_j$ defined by $g_{i,j}(x) = x^{2^{i-j}}$.

If α is a path in S between $(\ldots p_2, p_1, p_0)$ and $(\ldots 1, 1, 1)$, then $\alpha_i = \pi_i \circ \alpha$ is a path in C_i between 1 and p_i for all i. We note that α_i may have a winding number around C_i bigger than one. For this reason, let β_i be the shortest path from 1 to p_i in C_i. Then, the compositions $g_{i,0} \circ \pi_i \circ \alpha$ and $g_{i,0} \circ \beta_i$ are paths in C_0.



We compare their winding numbers. Let w be the winding number of a path in C_0, then for every n, we have $|2^n \frac{a(p_n)}{2\pi}| = |w(g_{n,0} \circ \beta_n)| \leq |w(g_{n,0} \circ \alpha_n)| = |w(\alpha_0)| \in \mathbb{R}$. Thus, for every n, $|2^n a(p_n)| \leq 2\pi |w(\alpha_0)|$, where the right hand side is a fixed real number. This proves one directional implication.

For the converse, suppose $\{|2^n a(p_n)|\}$ is bounded as n goes to infinity. Then, the limit of $\{|a(p_n)|\}$ is zero. Define the maps $b_n : \mathbb{R} \to C_n$ by $b_n(x) = e^{\frac{\pi i x}{2^{n-1}}}$. The space $\mathbb{R}$, along with the maps b_n, is a system compatible with the inverse system (C_n, π_n); thus, by the universal mapping property of inverse limits, there exists a unique map $\varprojlim b_n : \mathbb{R} \to S$ making the diagram commute.



However, the map $B : \mathbb{R} \to S$ defined by $B(x) = (\ldots, e^{\frac{\pi i x}{2^2}}, e^{\frac{\pi i x}{2}}, e^{\pi i x}, e^{2\pi i x)}$ obviously makes the diagram commute, so by uniqueness, $\varprojlim b_n = B$. Now, there exists $k \in \mathbb{N}$ such that for all natural numbers $m \geq k$, we have $|a(p_m)| < \frac{\pi}{2}$.

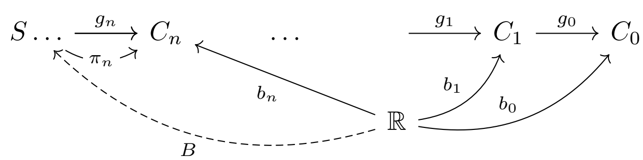
Case 1. $a(p_k) \geq 0$. We can write $p_k = e^{\pi i \theta_k}$ where $\frac{\pi}{2} > a(p_k) = \pi\theta_k \geq 0$. Normally we would not know p_{k+1}, since there are two possibilities it could be; however, the fact that $\frac{\pi}{2} > |a(p_{k+1})| \geq 0$ implies that $a(p_{k+1}) = \frac{\pi\theta_k}{2}$, so $a(p_{k+1}) = e^{\frac{\pi i \theta_k}{2}}$. In fact, inductively for $m \geq k$, if we know $a(p_m) = \pi\theta_m$, then we know that $a(p_{m+1}) = \frac{\pi\theta_m}{2}$. Working the other way, we know that $a(p_{k-1}) = 2a(p_k)$. Thus, $p_{k-1} = e^{\pi i 2\theta_k}$. By using the relation $p_i^2 = p_{i-1}$, we conclude that $p_0 = e^{\pi i 2^k \theta_k}$, and $(\ldots, p_{k+1}, p_k, \ldots, p_1, p_0) = (\ldots, e^{\frac{\pi i \theta_k}{2}}, e^{\pi i \theta_k}, \ldots, e^{\pi i 2^{k-1}\theta_k}, e^{\pi i 2^k \theta_k})$. Thus, if $x = 2^{k-1}$, then $B(x) = (\ldots, p_2, p_1, p_0)$.

Case 2. $a(p_k) < 0$. This case is similar to case 1 except we write $p_k = e^{\pi i \theta_k}$ where $\frac{-\pi}{2} < \pi\theta_k < 0$. As in case 1, we have $(\ldots, p_{k+1}, p_k, \ldots, p_1, p_0) = (\ldots, e^{\frac{\pi i \theta_k}{2}}, e^{\pi i \theta_k}, \ldots, e^{\pi i 2^{k-1}\theta_k}, e^{\pi i 2^k \theta_k})$, and if $x = 2^{k-1}$, then $B(x) = (\ldots, p_2, p_1, p_0)$.

This argument suffices to prove $(\ldots p_2, p_1, p_0)$ is in the same path component of S as the point $(\ldots 1, 1, 1)$. □

Proposition 2.1.1: There exists a continuous bijective function B from $\mathbb{R}$ to the path component of S containing the point $(\ldots 1, 1, 1)$.

Proof: As in the proof of Theorem 2.1.1, we define the maps $b_n : \mathbb{R} \to C_n$ by $b_n(x) = e^{\frac{\pi i x}{2^{n-1}}}$. Thus we get the unique map $B = (\ldots, e^{\frac{\pi i x}{2^2}}, e^{\frac{\pi i x}{2}}, e^{\pi i x}, e^{2\pi i x)}$ making the diagram commute.



We show B is injective. Suppose by way of contradiction, there exist real numbers $x \neq y$ such that $B(x) = B(y)$. Then, $B_0(x) = B_0(y)$ implies $e^{2\pi i x} = e^{2\pi i y}$ implying $|x - y| \in \mathbb{N}$. By factoring, we can write $|x - y| = 2^k m$ where k and m are integers, $k \geq 0$, $m > 0$, and $2 \nmid m$. In other words, $2^{k+1} \nmid |x - y|$. However, $B_{k+1}(x) = B_{k+1}(y)$ implies $e^{\frac{\pi i x}{2^k}} = e^{\frac{\pi i y}{2^k}}$, which implies $2^{k+1} \mid |x - y|$. This is a contradiction, and we conclude B is injective.

To show B is surjective onto the path component containing $(\ldots 1, 1, 1)$, let $(\ldots p_2, p_1, p_0)$ be a point in the path component. Then, by Theorem 2.1.1, we know the sequence $\{|a(p_n)|\}$ converges to zero as n goes to infinity. The argument used in the proof of the converse of Theorem 2.1.1 shows that there exists $x \in \mathbb{R}$ such that $B(x) = (\ldots p_2, p_1, p_0)$. Thus, B is surjective onto the path component containing $(\ldots, 1, 1, 1)$, and $B : \mathbb{R} \to S$ is a bijective map. □

We have the very useful criterion for determining when two points of S are in the same path component.

Corollary 2.1.1: Two points $(\ldots, b_1, b_0)$ and $(\ldots, c_1, c_0)$ in S are in the same path component if and only if the sequence $|2^n a(\frac{b_n}{c_n})|$ is bounded as n goes to infinity, where $a(\frac{b_n}{c_n}) \in [-\pi, \pi)$ is the argument of the complex number $\frac{b_n}{c_n}$.

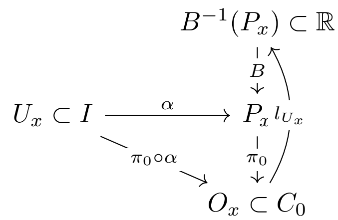
Proof: The points $(\ldots, b_1, b_0)$ and $(\ldots, c_1, c_0)$ are in the same path component if and only if the points $(\ldots, \frac{b_1}{c_1}, \frac{b_0}{c_0})$ and $(\ldots, 1, 1, 1)$ are in the same path component (because S is homogeneous); but $(\ldots, \frac{b_1}{c_1}, \frac{b_0}{c_0})$ and $(\ldots, 1, 1, 1)$ are in the same path component if and only if the sequence $\{|2^n a(\frac{b_n}{c_n})|\}$ is bounded as n goes to infinity. □

Corollary 2.1.2: There exists a bijective map from $\mathbb{R} \to L$ where L is any path component of S.

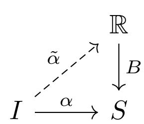
Proof: This follows from Proposition 2.1.1 and the fact that S is homogeneous. □

Proposition 2.1.2: Let I be any interval of $\mathbb{R}$ and $\alpha : I \to S$ be any map. Then α has a lift $\tilde{\alpha}$ to $\mathbb{R}$ such that $\alpha = B \circ \tilde{\alpha}$. Thus, $\mathbb{R}$ is a fibration over a path component of S, since any path in S can lift to a path in $\mathbb{R}$.

Proof: Since B is a bijection between $\mathbb{R}$ and a path component, L of S, then the inverse function $B^{-1} : L \to \mathbb{R}$ exists, though it is not continuous. The composition $\pi_0 \circ \alpha$ is a path in C_0. Fix $x \in I$ and let O_x be a connected open neighborhood ofC_0, small enough so as not to cover the circle C_0 but containing the point $\pi_0(\alpha(0))$. The preimage of O_x under π_0 is open by continuity, so we can take a neighborhood W_x of $\alpha(x)$ so that the path component, P_x, of W_x containing $\alpha(x)$ satisfies $\pi_0(P_x) = O_x$. Now, B is continuous, so the preimage of W_x is open $\mathbb{R}$, thus it is a union of open intervals. The point $B^{-1}(\alpha(x))$ is in one of these intervals, call it V_x. B restricted to the closure of V_x is a homeomorphism onto its image, therefore $B(V_x) = P_x$, or put another way, $V_x = B^{-1}(P_x)$, and P_x is the homeomorphic image of the open interval V_x. The preimage of W_x is open in I by continuity, so there exists a connected open neighborhood U_x about x so that $\alpha(U_x) \subset W_x$. However, since U_x is connected, and α is continuous, we have the stronger result $\alpha(U_x) \subset P_x$. We now define the map $l_{U_x} : O_x \subset C_0 \to B^{-1}(P_x)$ so that for each $w \in P_x$ we have $l_{U_x}(\pi_0(w)) = B^{-1}(w)$. We then define the map $\tilde{\alpha}_{U_x} : U_x \subset I \to B^{-1}(P_x)$ by $\tilde{\alpha}_{U_x} = l_{U_x} \circ \pi_0 \circ \alpha|_{U_x}$.



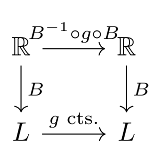
If two maps $\tilde{\alpha}_{U_x}$ and $\tilde{\alpha}_{U_y}$ have overlapping domains U_x and U_y, so that $z \in U_x \cap U_y$, then $\tilde{\alpha}_{U_x}(z) = l_{U_x}(\pi_0(\alpha(z))) = B^{-1}(\alpha(z)) = \tilde{\alpha}_{U_y}(z)$. Thus the two maps agree on their overlapping domains. We then have a well-defined map $\tilde{\alpha} : I \to \mathbb{R}$ defined for $x \in I$ by $\tilde{\alpha}(x) = \tilde{\alpha_{U_x}}(x)$. To conclude the proof, we observe that for $x \in I$, we have $(B \circ \tilde{\alpha})(x) = B(\tilde{\alpha}(x)) = B(\tilde{\alpha_{U_x}}(x)) = B(B^{-1}(\alpha(x))) = \alpha(x)$.



□

Proposition 2.1.3: Let L denote a path component of S. If $g : L \to L$ is continuous, then $B^{-1} \circ g \circ B : \mathbb{R} \to \mathbb{R}$ is continuous.

Proof: We have the following diagram



To see that $B^{-1} \circ g \circ B : \mathbb{R} \to \mathbb{R}$ is continuous, let x be an image point in $\mathbb{R}$, and let t be in the pre-image of x. Choose a connected open interval V_x with length less than one containing x. Since B restricted to the closure of V_x is a homeomorphism onto the image $B(cl(V_x))$ with the subspace topology, we have $B(V_x) = P_x$ is a connected open set in $B(cl(V_x))$. By definition of the subspace topology, $P_x = W_x \cap B(cl(V_x))$ for an open set W_x of S; therefore P_x is a path component of W_x. The composition $g \circ B$ is continuous, so the preimage of W_x under this composition is open in $\mathbb{R}$, thus it is a union of open intervals. One of them contains t, so we can choose a small connected neighborhood U_x containing t so that $(g \circ B)(U_x) \subset W_x$. But since U_x is connected and $g \circ B$ is continuous, we have the stronger result that $(g \circ B)(U_x) = g(B(U_x)) \subset P_x$. This implies that $B^{-1}(g(B(U_x))) \subset B^{-1}(P_x) = V_x$. This is sufficient to conclude the proof. □

2.2 The Solenoid is a Sharkovskii Space

We have the following theorem.

Theorem 2.2.1: If g is a continuous function from a path component L, of S to itself, then g has the Sharkovskii Property. Therefore, L is a Sharkovskii Space.

Proof: The following computation shows that $B^{-1} \circ g \circ B$ has an n-orbit if and only if g has an n-orbit.

$$
\begin{aligned}
&(B^{-1} \circ g \circ B)^n(x) = x \\
\Longleftrightarrow &(B^{-1} \circ g^n \circ B)(x) = x \\
\Longleftrightarrow &(g^n \circ B)(x) = B(x) \\
\Longleftrightarrow &g^n(B(x)) = B(x)
\end{aligned}
$$

Since $B^{-1} \circ g \circ B$ is a continuous map from $\mathbb{R}$ to $\mathbb{R}$, it has the Sharkovskii property. However, the previous computation shows that g must also have the Sharkovskii Property. Since g was an arbitrary map from L to itself, we conclude that L is a Sharkovskii Space. $\square$

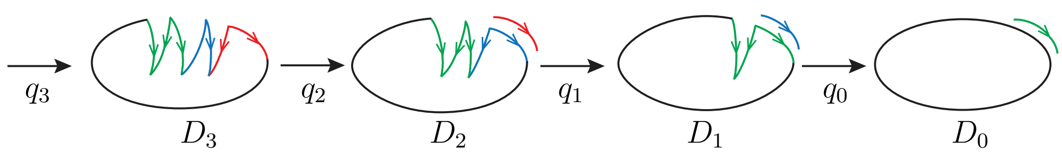
Chapter 3. The Warsawanoid

3.1 The Warsaw Circle

The Solenoid is the inverse limit of circles with connecting maps $g_i(x) = x^2$. If we define new connecting maps between circles, we will have a new inverse system with a different inverse limit.

Definition 3.1.1 For each $i \in \mathbb{N} \cup \{0\}$, let C_i be the unit circle in the complex plane parametrized by angle. Define the maps $f_i : C_{i+1} \to C_i$ to be the following quotient map $f(C_{i+1}) = \frac{C_i}{\frac{2\pi}{3}+\epsilon=\frac{2\pi}{3}-\epsilon \text{ and } \frac{\pi}{3}+\epsilon=\frac{\pi}{3}-\epsilon}$ for $\epsilon \in [0, \frac{\pi}{3}]$. The inverse limit of the inverse system (C_i) with maps (f_i) we call the *Warsaw Circle.*

This quotient map can have the following geometric interpretation.



This motivates the next section.

3.2 A 2 by 2 Diagram of Inverse Limits and the Warsawanoid

Definition 3.2.1: For $i, j \in \mathbb{N} \cup \{0\}$, let $C_{i,j}$ be the unit circle in the complex plane. Define the maps $g_{i,j} : C_{i+1,j} \to C_{i,j}$ by $g_{i,j}(x) = x^2$ so that $C_{i+1,j}$ is a double cover of $C_{i,j}$ with covering map $g_{i,j}$. Also define $h_{0,j} : C_{0,j+1} \to C_{0,j}$ to be the quotient map f from definition 3.1.1

Recall the fact that every map between spaces induces a map between the fundamental groups of the spaces.

Proposition 3.2.1: Let $f : C_{0,1} \to C_{0,0}$ be the quotient map given in definition 3.2.1. Then the induced map $f_* : \pi_1(C_{0,1}) \to \pi_1(C_{0,0})$ from the fundamental group of $C_{0,1}$ to the fundamental group of $C_{0,0}$ is an isomorphism.

Proof: The fundamental groups of both spaces are isomorphic to $\mathbb{Z}$, and f_* maps the generator of $\pi_1(C_{0,1})$ to the generator of $\pi_1(C_{0,0})$. It is a basic fact that f_* is an isomorphism. $\square$

Proposition 3.2.2: Let $h_{0,j} : C_{0,j+1} \to C_{0,j}$ be the map defined in definition 3.2.1; the map $h_{0,j} \circ g_{0,j+1} : C_{1,j+1} \to C_{0,j}$ lifts to a map $h_{1,j} : C_{1,j+1} \to C_{1,j}$ such that $g_{0,j} \circ h_{1,j} = h_{0,j} \circ g_{0,j+1}$.

Proof: We use the Lifting Criterion [19, 61-62]. The composition $h_{0,j} \circ g_{0,j+1}$ is a map from $C_{1,j+1}$ to $C_{0,j}$, so it has an induced map between $\pi_1(C_{1,j+1})$ and $\pi_1(C_{0,j})$, both of which are isomorphic to $\mathbb{Z}$. The image of $\pi_1(C_{1,j+1})$ in $\pi_1(C_{0,j})$ under the map is isomorphic to $2\mathbb{Z}$ because of the map $g_{0,j+1}$. However, the map $g_{0,j} : C_{1,j} \to C_{0,j}$ double covers $C_{0,j}$ making $C_{1,j}$ a covering space of $C_{0,j}$. Again, the image of $\pi_1(C_{1,j})$ under the induced homomorphism is isomorphic to $2\mathbb{Z}$. Therefore, by the lifting criterion, we can lift the map $h_{0,j} \circ g_{0,j+1} : C_{1,j+1} \to C_{0,j}$ to a map, $h_{1,j} : C_{1,j+1} \to C_{1,j}$ such that $g_{0,j} \circ h_{1,j} = h_{0,j} \circ g_{0,j+1}$. $\square$

In fact, we can inductively define $h_{i+1,j} : C_{i+1,j+1} \to C_{i+1,j}$ such that $g_{i,j} \circ h_{i+1,j} = h_{i,j} \circ g_{i,j+1}$. Here's the inductive construction. If $h_{i,j} : C_{i,j+1} \to C_{i,j}$ is given, define $h_{i+1,j} : C_{i+1,j+1} \to C_{i+1,j}$ to be the lift of $h_{i,j} \circ g_{i,j+1}$ such that $g_{i,j} \circ h_{i+1,j} = h_{i,j} \circ g_{i,j+1}$. Such a lift exists by the Lifting Criterion, since the image of $\pi_1(C_{i+1,j+1})$ in $\pi_1(C_{i,j})$ under the induced homomorphism is isomorphic to $2\mathbb{Z}$ which is contained in the image of $\pi_1(C_{i+1,j})$, also isomorphic to $2\mathbb{Z}$, in $\pi_1(C_{i,j})$ under the homomorphism induced from the covering map $g_{i,j}$.

Definition 3.2.2: Define $h_{i+1,j} : C_{i+1,j+1} \to C_{i+1,j}$ to be the lift of $h_{i,j} \circ g_{i,j+1} : C_{i+1,j+1} \to C_{i,j}$ making $g_{i,j} \circ h_{i+1,j} = h_{i,j} \circ g_{i,j+1}$.

This gives us the following commutative diagram for all i, j.

$$\begin{array}{ccc} C_{i+1,j+1} & \xrightarrow{h_{i+1,j}} & C_{i+1,j} \\ \downarrow{\scriptstyle g_{i,j+1}} & & \downarrow{\scriptstyle g_{i,j}} \\ C_{i,j+1} & \xrightarrow{h_{i,j}} & C_{i,j} \end{array}$$

Definition 3.2.3: For fixed j, the inverse limit of the system $(C_{i,j})$ with maps $(g_{i,j})$ is a solenoid which we call S_j with projection maps $\pi_{i,j} : S_j \to C_{i,j}$. For fixed i, the inverse limit of the system $(C_{i,j})$ with maps $(h_{i,j})$ is a Warsaw Circle which we call WC_i with projection maps $P_{i,j} : WC_i \to C_{i,j}$.

The universal mapping property of inverse limits enables us to make the following definition.

Definition 3.2.4: Fix j. Let $H_j : S_{j+1} \to S_j = \varprojlim(h_{i,j} \circ \pi_{i,j+1})$ be the unique map such that $\pi_{i,j} \circ H_j = h_{i,j} \circ \pi_{i,j+1}$. Similarly, for fixed i, let $G_i : WC_{i+1} \to WC_i = \varprojlim(g_{i,j} \circ P_{i+1,j})$ be the unique map such that $g_{i,j} \circ P_{i+1,j} = P_{i,j} \circ G_i$.

Definition 3.2.5: The inverse system (WC_i) with connecting maps (G_i) has as its inverse limit *The Warsawanoid*, denoted by W, with projection maps $\Gamma_i : W \to WC_i$.

Fact: [20, 72] Inverse limits commute, which is another way of saying that the Warsawanoid is also the inverse limit of the inverse system (S_j) with maps (H_j).

Definition: 3.2.6: The Warsawanoid is the inverse limit of the inverse system $(S_j), (H_j)$. The projection maps are $\eta_j : W \to S_J$.

Proposition 2.1.1 from section 2 assures the existence of a continuous bijection between $\mathbb{R}$ and any path component of a solenoid. Therefore, we give the following definition.

Definition 3.2.7: Denote by L_j a copy of $\mathbb{R}$ equipped with a continuous bijection $B_j : \mathbb{R} \to L$ where L is a path component of the solenoid S_j. We remind the reader that L_j is a fibration over L, that is, paths in L can be lifted to paths in L_j.

The composition $H_j \circ B_{j+1} : L_{j+1} \to S_j$ is a map from $\mathbb{R}$ to a path component of S_j. By proposition 2.1.2, it has a lift, $H_j : L_{j+1} \to L_j$ such that $B_j \circ H_j = H_j \circ B_{j+1}$.

Definition 3.2.8: Let $\gamma_j : L_{j+1} \to L_j$ be the map promised by proposition 2.1.2 such that $B_j \circ \gamma_j = H_j \circ B_{j+1}$. Further, we define the inverse limit of the inverse system (L_j) with connecting maps (γ_j) to be the Warsaw Line denoted by L_ω. The projection maps are $\beta_j : L_\omega \to L_j$.

3.3 Properties of the Warsawanoid

For a path component K_0, of S_0, it has a unique preimage path component K_1 in S_1 under H_0. Inductively, if K_j is the unique preimage path component in S_j of K_{j-1} in S_{j-1} under H_{j-1}, then let K_{j+1} be the unique preimage path component in S_{j+1} of K_j under H_j. The path components K_j with connecting maps H_j motivate the next definition.

Definition 3.3.1: We define a *Warsawanoid Leaf* L' to be the subspace of coherent sequences $(\ldots, x_1, x_0) \in W$ such that for all j, $x_j \in K_j$, where K_j is the unique path component of S_j satisfying $H_{j-1}(K_j) = K_{j-1}$ for path component $K_{j-1} \subset S_{j-1}$.

Proposition 3.3.1: There exists a continuous bijection, $\tilde{B}$, between the Warsaw Line L_ω and L'.

Proof: The maps $B_j \circ \beta_j$ are compatible maps from L_ω to S_j. By the universal mapping property, we obtain a unique map $\tilde{B} = \varprojlim(B_j \circ \beta_j)$ from L_ω to W making all diagrams commute. This map is $\tilde{B}(...x_2, x_1, x_0) = (..., B_2(x_2), B_1(x_1), B_0(x_0))$, for all coherent sequences $(x_i) \in L_\omega$. To show this is indeed the unique map promised by the universal mapping property, observe that B_j is a bijection, so B_j^{-1} exists. Thus, $H_j(B_{j+1}(x_{j+1})) = (B_j \circ H_j \circ B_{j+1}^{-1})(B_{j+1}(x_{j+1})) = B_j(H_j(x_{j+1}) = B_j(x_j)$.

$$\begin{array}{ccc} L_{j+1} & \xrightarrow{\gamma_j} & L_J \\ \downarrow{\scriptstyle B_{j+1}} & & \downarrow{\scriptstyle B_j} \\ K_{j+1} & \xrightarrow{H_j} & K_j \end{array}$$

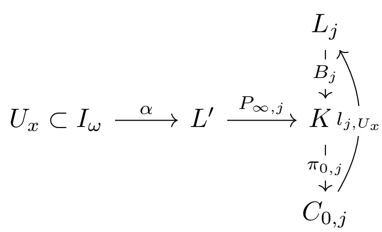
To show $\tilde{B}$ is injective, suppose $(\ldots, x_2, x_1, x_0) \neq (\ldots, y_2, y_1, y_0)$ are coherent sequences in L_ω. Then, for some n we have $x_n \neq y_n$, implying $B_n(x_n) \neq B_n(y_n)$ since B_n is injective. Thus, $(\ldots, B_1(x_1), B_0(x_0)) \neq (\ldots, B_1(y_1), B_0(y_0))$, since $B_n(x_n) \neq B_n(y_n)$. By definition of $\tilde{B}$ we have $\tilde{B}(\ldots, x_2, x_1, x_0) \neq \tilde{B}(\ldots, y_2, y_1, y_0)$.

To show $\tilde{B}$ is surjective, suppose $(\ldots, x_2, x_1, x_0)$ is an element of L', so it is a coherent sequence. Then because of the commutativity of the diagram above, the sequence $(\ldots, B^{-1}(x_2), B^{-1}(x_1), B^{-1}(x_0))$ is also coherent, so it is a point in L_ω, and it is the preimage of $(\ldots, x_2, x_1, x_0)$ under $\tilde{B}$. Therefore $\tilde{B}^{-1}$ exists, and has formula $\tilde{B}^{-1}(\ldots, x_2, x_1, x_0) = (\ldots, B_2^{-1}(x_2), B_1^{-1}(x_1), B_0^{-1}(x_0))$ for $(\ldots, x_2, x_1, x_0) \in L'$. This concludes the proof. □

Definition 3.3.2: Denote by I_ω The topologist's sine curve.

Proposition 3.3.2: If α is a map from I_ω to L', then there exists a lift, $\tilde{\alpha} : I_\omega \to L_\omega$ such that $\alpha = \tilde{B} \circ \tilde{\alpha}$. This proposition shows there exists "I_ω"-lifting for L', analogous to path-lifting for a path component of the solenoid.

Proof: Let $x \in L_\omega$. We have $\alpha(x) \in L'$, so we can write $\alpha(x)$ as a coherent sequence $(\ldots, k_1, k_0)$. Therefore, $\eta_j(\alpha(x)) = k_j \in S_j$. The composition $\pi_{0,j} \circ \eta_j \circ \alpha$ is a map from I_ω to $C_{0,j}$. Let O_x be a connected open set of $C_{0,j}$ that contains $\pi_{0,j}(\eta_j(\alpha(x)) = \pi_{0,j}(k_j))$ such that O_x does not cover all of $C_{0,j}$. By continuity, we can take an open neighborhood W_x of S_j so that P_x, the path component of W_x containing k_j satisfies $\pi_{0,j}(P_x) = O_x$. Since W_x is open in S_j and B_j is continuous, its preimage is open and so is a union of open intervals in $\mathbb{R}$. Let V_x be the one containing $B_j^{-1}(k_j)$ so that $B_j(V_x) = P_x$, or equivalently, $V_x = B_j^{-1}(P_x)$. We define a map $l_{U_x,j} : O_x \to V_x$ where for $z \in P_x$, we have $l_{U_x,j}(\pi_{0,j}(z)) = B_j^{-1}(z)$. We observe that $(B_j \circ (l_{U_x,j} \circ \pi_{0,j})(z) = B_j(l_{U_x,j}(\pi_{0,j}(z))) = B_j(B_j^{-1}(z)) = z$. Then, the function $\tilde{\alpha}_{U_x,j} = l_{U_x,j} \circ \pi_{0,j} \circ \eta_j \circ \alpha|_{U_x}$ is continuous from a connected open neighborhood U_x of $x \in I_\omega$ to $V_x \subset L_j$. The following detail must be mentioned: We can only choose a connected open neighborhood of x if x does not lie on a vertical line in L_ω. However, if it does, we can take U_x to be the vertical path component of the neighborhood of x.



Suppose $\tilde{\alpha}_{U_x,j}$ and $\tilde{\alpha}_{U_y,j}$ have overlapping domains, so that $w \in U_x \cap U_y$ where U_x and U_y are open sets in I_ω. Then, $\tilde{\alpha}_{U_x,j}(w) = (l_{U_x,j} \circ \pi_{0,j} \circ \eta_j \circ \alpha|_{U_x})(w) = l_{U_x,j}(\pi_{0,j}(\eta_j(\alpha(w)))) = B^{-1}(\eta_j(\alpha(w))) = l_{U_y,j}(\pi_{0,j}(\eta_j(\alpha(w)))) = (l_{U_y,j} \circ \pi_{0,j} \circ \eta_j \circ \alpha|_{U_y})(w) = \tilde{\alpha}_{U_y,j}(w)$. Thus, $\tilde{\alpha}_{U_x,j}$ and $\tilde{\alpha}_{U_y,j}$ agree on their overlapping domains. We define the map $\tilde{\alpha}_j(x) : I_\omega \to L_j$ by $\tilde{\alpha}_j(x) = \tilde{\alpha}_{U_x,j}(x)$, which is well-defined for all $x \in I_\omega$. Fix $x \in I_\omega$, we can write $\alpha(x) \in L'$ as a coherent sequence $(\ldots, k_1, k_0)$, so that $\eta_j(\alpha(x)) = k_j$. Now observe the following computation: $(B_j \circ \tilde{\alpha}_j)(x) = B_j(\tilde{\alpha}_j(x)) = B_j(\tilde{\alpha}_{U_x,j}(x)) = B_j(l_{U_x,j}(\pi_{0,j}(\eta_j(\alpha(x))))) = B_j(B_j^{-1}(\eta_j(\alpha(x)))) = \eta_j(\alpha(x)) = (\eta_j \circ \alpha)(x) = k_j$. Thus the following diagram commutes

$$\begin{array}{ccc} & & L_j \\ & \overset{\tilde{\alpha}_j}{\nearrow} & \downarrow B_j \\ I_\omega & \xrightarrow{\eta_j \circ \alpha} & K_j \end{array}$$

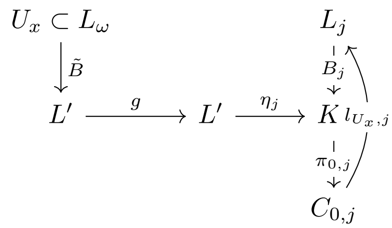
However, recall the following commutative diagram.

$$\begin{array}{ccc} L_{j+1} & \xrightarrow{\gamma_j} & L_J \\ \downarrow B_{j+1} & & \downarrow B_j \\ K_{j+1} & \xrightarrow{H_j} & K_j \end{array}$$

The commutativity of these two diagrams imply $\tilde{\alpha}_j = \gamma_j \circ \tilde{\alpha}_{j+1}$. Thus, the space I_ω with the maps $\tilde{\alpha}_j$ are compatible with the inverse system (L_j, γ_j). So by the universal mapping property, we get a unique map $\tilde{\alpha} = \varprojlim \tilde{\alpha}_j : I_\omega \to L_\omega$ with formula $\tilde{\alpha}(x) = (\ldots, \tilde{\alpha}_1(x), \tilde{\alpha}_0(x))$. We are at last able to show that $\alpha = \tilde{B} \circ \tilde{\alpha}$. Let $x \in I_\omega$ be arbitrary. Then, $\alpha(x) \in L'$, so $\alpha(x) = (\ldots, k_1, k_0)$ where $H_j(k_{j+1}) = k_j$. But we showed above that $(B_j \circ \tilde{\alpha}_j)(x) = B_j(\tilde{\alpha}_j(x)) = k_j$ for all j. Therefore, $(\tilde{B} \circ \tilde{\alpha})(x) = \tilde{B}(\tilde{\alpha}(x)) = \tilde{B}(\ldots, \tilde{\alpha}_1(x), \tilde{\alpha}_0(x)) = (\ldots, B_1(\tilde{\alpha}_1(x)), B_0(\tilde{\alpha}_0(x))) = (\ldots, k_1, k_0) = \alpha(x)$. Since $x \in I_\omega$ was arbitrary, we conclude that $\alpha = \tilde{B} \circ \tilde{\alpha}$. □

Proposition 3.3.3: If $g : L' \to L'$ is continuous, then $\tilde{B}^{-1} \circ g \circ \tilde{B} : L_\omega \to L_\omega$ is continuous.

Proof: Fix j, and let $x \in L_\omega$ be an arbitrary fixed point. We denote $g(\tilde{B}(x)) \in L'$ by the coherent sequence $(\ldots, k_1, k_0)$, so that $\eta_j(g(\tilde{B}(x))) = \eta_j((\ldots, k_1, k_0)) = k_j$. We let $O_x \subset C_{0,j}$ be a connected neighborhood of $\pi_{(0,j)}(k_j)$ not covering all of $C_{0,j}$. Continuity lets us find a neighborhood W_x in S_j of k_j so that P_x is the path component of W_x containing k_j where $\pi_{0,j}(P_x) = O_x$. By arguments similar to the ones used in the proofs of propositions 2.1.2 and 3.2.2, we know there exists a connected open interval $V_x \subset L_j$ so that $V_x = B_j^{-1}(P_x)$. We again define maps $l_{U_x,j} : O_x \to B_j^{-1}(P_x)$ so that for $w \in P_x$, we have $l_{U_x,j}(\pi_{0,j}(w)) = B_j^{-1}(w)$, and the composition functions $\varphi_{U_x,j} = l_{U_x,j} \circ \pi_{0,j} \circ \eta_j \circ g \circ \tilde{B}|_{U_x}$ agree whenever their domains (connected open neighborhoods U_x and U_y, or the vertical path components of said neighborhoods) overlap.



Thus we get a well-defined map $\varphi_j : L_\omega \to L_j$ defined by $\varphi_j(x) = \varphi_{U_x,j}(x)$. We also have $\varphi_j(x) = (l_{U_x,j} \circ \pi_{0,j} \circ \eta_j \circ g \circ \tilde{B})(x) = (B_j^{-1} \circ \eta_j \circ g \circ \tilde{B})(x) = B_j^{-1}(\eta_j(g(\tilde{B}(x)))) = B_j^{-1}(k_j)$, where k_j is the j-th component of the coherent sequence $g(\tilde{B}(x)) \in L'$.

$$\begin{array}{ccccc} L_\omega & & \xrightarrow{\gamma_j = B_j^{-1} \circ \eta_j \circ g \circ \tilde{B}} & & L_j \\ \Big\downarrow \tilde{B} & & & & \Big\uparrow B_j^{-1} \\ L' & \xrightarrow{g} & L' & \xrightarrow{\eta_j} & K_j \end{array}$$

We must show that L_ω along with the maps φ_j are compatible with the inverse system (L_ω, γ_j). To this end, let $x \in L_\omega$ be arbitrary and observe that $g(\tilde{B}(x)) \in L'$, so we can write $g(\tilde{B}(x)) = (\ldots, k_1, k_0)$ where $H_j(k_{j+1}) = k_j$ for all j. Also, we note that $\eta_j(g(\tilde{B}(x))) = \eta_j(\ldots, k_1, k_0) = k_j$.

We have the following diagram

$$\begin{array}{ccc} L_{j+1} & \xrightarrow{\gamma_j} & L_J \\ \Big\downarrow B_{j+1} & & \Big\downarrow B_j \\ K_{j+1} & \xrightarrow{H_j} & K_j \end{array}$$

Then, $B_j(\gamma_j(B_{j+1}^{-1}(k_{j+1}))) = H_j(k_{j+1}) = k_j$ implying $\gamma_j(B_{j+1}^{-1}(k_{j+1})) = B_j^{-1}(k_j)$, which is equivalent to $\gamma_j(\varphi_{j+1}(x)) = \varphi_j(x)$. So L_ω along with the maps φ_j are compatible with the inverse system (L_j, γ_j), so be the universal mapping property, we get a unique map $\phi = \varprojlim \varphi_j : L_\omega \to L_\omega$ given by the formula $\phi(x) = (\ldots, \varphi_1(x), \varphi_0(x))$, where we suppress the sequence notation for $x \in L_\omega$. We show that $\phi = \tilde{B}^{-1} \circ g \circ \tilde{B}$. Fix $x \in L_\omega$, we again write $g(\tilde{B}(x)) = (\ldots, k_1, k_0)$, so that $\eta_j(g(\tilde{B}(x))) = k_j$. Then, $(\tilde{B}^{-1} \circ g \circ$

$\tilde{B})(x) = \tilde{B}^{-1}(g(\tilde{B}(x))) = \tilde{B}^{-1}((\ldots, k_1, k_0)) = (\ldots, B_1^{-1}(k_1), B_0^{-1}(k_0))$; however $\phi(x) = (\ldots, \varphi_1(x), \varphi_0(x)) = (\ldots, B_1^{-1}(\eta_1(g(\tilde{B}(x))))), B_0^{-1}(\eta_0(g(\tilde{B}(x))))) = (\ldots, B_1^{-1}(k_1), B_0^{-1}(k_0))$. This completes the proof. □

3.4 The Warsawanoid is a Sharkovskii Space

We have the following theorem.

Theorem 2 If $g : L' \to L'$ is continuous, then it has the Sharkovskii Property. Therefore, L' is a Sharkovskii Space.

Proof: L_ω is a Sharkovskii Space [21]. Thus, the map $\tilde{B}^{-1} \circ g \circ \tilde{B} : L_\omega \to L_\omega$ has the Sharkovskii Property.

$$\begin{array}{ccc} L_\omega & \xrightarrow{\tilde{B}^{-1}\circ g\circ\tilde{B}} & L_\omega \\ \downarrow{\scriptstyle \tilde{B}} & & \downarrow{\scriptstyle \tilde{B}} \\ L' & \xrightarrow{g} & L' \end{array}$$

The orbits of $\tilde{B}^{-1} \circ g \circ \tilde{B}$ are in one-to-one correspondence with the orbits of g, as in the proof of Theorem 1.2. Thus, g also has the Sharkovskii Property, and L' is a Sharkovskii Space. □

Bibliography

[1] Lawaetz (*The Sharkovski Theorem); Aktuel Matematik, 2008*

*[2] Li Yorke (*Period 3 Implies Chaos), The Mathematical Association of America, The American Mathematical Monthly, Vol. 82, No. 10, Dec. 1975

[3] Nitecki Robinson (*Global Theory of Dynamical Systems), Springer; New York, NY, Oct. 2006*

*[4] Ciesielski Pogoda (*On Ordering the Natural Number, or, The Sharkovskii Theorem), The Mathematical Association of America, The American Mathematical Monthly, Vol. 115, No. 2, Feb. 2008

[5] Misiurewicz (*Periodic points of maps of degree one of a circle), Ergodic Theory Dynamical Systems, 2 (1982), 221227*

*[6] Siegberg (*Chaotic mappings on S^1, periods one, two, three imply chaos on S^1), Numerical Solution of Nonlinear Equations (Bremen, 1980), Lecture Notes in Math., 878, Springer, Berlin-New York, (1981), 351370

[7] Alseda Soler (*Periodic points of continuous mappings of the circle), Publ. Sec. Univ. Autonoma Barcelona, (1981), 571*

*[8] Baldwin (*An extension of Sarkovskis theorem to the n-od), Ergodic Theory Systems, 11 (1991), 249271

[9] Alseda Llibre Misiurewicz (*Periodic orbits of maps of Y), Amer. Math. Soc., 313 (1989), 475538*

*[10] Demin (*Coexistence of periodic, almost periodic and recurrent points of transformations of n-od), Vestnik Moskov. Univ. Ser. I Mat. Mekh., 1996, 8487; translation in Moscow

Univ Math. Bull., 51 (1996), 4648

[11] Alseda Juher Mumbru (*Sets of periods for piece-wise monotone tree maps), Internat. J. Bifur. Chaos Appl. Sci. Engrg., 13 (2003), 311341*

*[12] Baldwin (*Versions of Sharkovskis theorem on trees and dendrites), Topology Proc. 18 (1993), 1931

[13] Gillot Llibre (*Periods for maps of the figure-eight space), Reprint of the paper reviewed in MR1361924 (97d:58161), World Sci. Ser. Nonlinear Sci. Ser. B Spec. Theme Issues Proc., 8, in Thirty Years After Sharkovskis Theorem: New Perspectives (Murcia, 1994), World Scientific Publ., River Edge, NJ, (1995), 95106*

*[14] Jin Cheng Xiong, Xiang Dong Ye, Zhi Qiang Zhang, Jun Huang (*Some dynamical properties of continuous maps on the Warsaw circle), Acta Math. Sinica (Chin. Ser.), 39 (1996), 294299

[15] Li Zhen Zhou and You Cheng Zhou, (*Some dynamical properties of continuous self-maps on the k-Warsaw circle), J. Zhejiang Univ. Sci. Ed., 29 (2002), 1216*

*[16] W. T. Ingram (*Periodic points for homeomorphisms of hereditarily decomposable chainable continua), Proc. Amer. Math. Soc., 107 (1989), 549553

[17] Burns Hasselblatt (*The Sharkovsky Theorem: A Natural Direct Proof), The Mathematical Association of America, The American Mathematical Monthly, 118 (3), 2011*

*[18] Spanier (*Algebraic Topology), Springer-Verlag; New York, NY 1966

[19] Hatcher (*Algebraic Topology), Cambridge University Press; New York, NY 2001*

*[20] Borceux (*Handbook of Categorical Algebra Vol 1), Cambridge University Press; Cambridge, Great Britain 1994

[21] Conner Grant Meilstrup (*A Sharkovsky Theorem for non-locally Connected Spaces), Discrete and Continuous Dynamical Systems, Volume 32, Number 10, October 2012*